Mon. Not. R. Astron. Soc. **000**, 000–000 (0000) Printed 2 February 2008 (MN LaTeX style file v1.4)

Axisymmetric stability criterion for two gravitationally coupled singular isothermal discs

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Accepted 2003 ... Received 2003 ...; in original form 2003 ...

ABSTRACT

Using the two-fluid formalism with the polytropic approximation, we examine the axisymmetric stability criterion for a composite system of gravitationally coupled stellar and gaseous singular isothermal discs (SIDs). Both SIDs are taken to be infinitely thin and are in a self-consistent steady background rotational equilibrium with power-law surface mass densities ($\propto r^{-1}$) and flat rotation curves. Recently, Lou & Shen (2003) derived exact solutions for both axisymmetric and nonaxisymmetric stationary perturbations in such a composite SID system and proposed the D_s-criterion of stability for axisymmetric perturbations. Here for axisymmetric perturbations, we derive and analyze the time-dependent WKBJ dispersion relation to study stability properties. By introducing a dimensionless stellar SID rotation parameter D_s, defined as the ratio of the constant stellar rotation speed V_s to the constant stellar velocity dispersion a_s, one can readily determine the axisymmetric stability D_s-criterion numerically by identifying a stable range of D_s. Those systems which rotate too slow (collapse) or too fast (ring fragmentation) are unstable. We found that the stable range of D_s^2 depends on the mass ratio δ of the gaseous SID to the stellar SID and on the square of the ratio β of the stellar velocity dispersion (which mimics the sound speed) to the gaseous isothermal sound speed. Increment of either δ or β or both will diminish the stable range of D_s^2. The WKBJ results of instabilities provide physical explanations for the stationary configurations derived by Lou & Shen. It is feasible to introduce an effective Q parameter for a composite SID system. The closely relevant theoretical studies of Elmegreen (1995), Jog (1996) and Shu et al. (2000) are discussed. A study of composite partial SID system reveals that an axisymmetric dark matter halo will promote stability of composite SID system against axisymmetric disturbances. Potential applications to disc galaxies, circumnucleus discs around nuclei of galaxies, and protostellar discs are briefly discussed.

Key words: stars: formation—ISM: general—galaxies: kinematics and dynamics—galaxies: spiral—galaxies: structure.

1 INTRODUCTION

In the context of galactic disc dynamics, there have been numerous investigations on criteria of axisymmetric and non-axisymmetric disc instabilities (e.g., Binney & Tremaine 1987). For a single disc of either gaseous or stellar content, Safronov (1960) and Toomre (1964) originally derived a dimensionless Q parameter to determine the local stability (i.e., $Q > 1$) against axisymmetric ring-like disturbances. Besides a massive dark matter halo, a more realistic disc galaxy involves both gas and stars. It is thus sensible to consider a composite system of one gas disc and one stellar disc under the gravitational control of the dark matter halo. Theoretical studies on this type of two-component disc systems were extensive in the past (Lin & Shu 1966, 1968; Kato 1972; Jog & Solomon 1984a, b; Bertin & Romeo 1988; Romeo 1992; Elmegreen 1995; Jog 1996; Lou & Fan 1998b, 2000a, b). In particular, it has been attempted to introduce proper definitions of an effective Q parameter relevant to a composite disc system for the criterion of local axisymmetric instabilities (Elmegreen 1995; Jog 1996; Lou & Fan 1998b). The results of such a stability analysis may provide a basis for understanding composite disc dynamics (e.g. Lou & Fan 2000a, b; Lou & Shen 2003) and for estimating global star

formation rate in a disc galaxy (e.g., Kennicutt 1989; Silk 1997).

There are several reasons that lead us to once again look into this axisymmetric stability problem. A few years ago, Shu, Laughlin, Lizano & Galli (2000; see also Galli et al. 2001) studied stationary coplanar perturbation structures in an isopedically magnetized singular isothermal disc (SID) without using the usual WKBJ approximation. They found exact analytical solutions for stationary (i.e., zero pattern speed) configurations of axisymmetric and non-axisymmetric logarithmic spiral perturbations. According to their analysis, for axisymmetric perturbations with radial propagation, a SID with sufficiently slow rotation speed will collapse for sufficiently large radial perturbation scales, while a SID with sufficiently fast rotation speed will be unstable to ring fragmentations for relatively small radial perturbation scales (see Fig. 2 of Shu et al. 2000). For axisymmetric perturbations, the stable regime of SID rotation speed may be characterized by a dimensionless rotation parameter D, which is the ratio of the constant SID rotation speed V to the isothermal sound speed a. The critical values of the lowest and the highest D for axisymmetric stability can be derived from their marginal stability curves. Shu et al. (2000) supported their interpretations by invoking the familiar Toomre Q parameter, implying that a local WKBJ analysis may still have some relevance or validity to global SID perturbation solutions. In their proposed scheme, Shu et al. noted that the two critical values of D are fairly close to $Q = 1$ for neutral stability. As they did not invoke the WKBJ approximation in deriving the analytic perturbation solutions, the $Q-$criterion and the $D-$criterion correspond to each other well with large wavenumber (ring fragmentation), while the rough correspondence of the $Q-$criterion and the $D-$criterion for small wavenumbers (collapse) is unexpected where the WKBJ requirement is poorly met.

Regarding astrophysical applications, these basic properties of $D-$criterion should be properly applied in contexts even at qualitative or conceptual levels. For example, for majority of currently observed disc galaxies, the typical rotation parameter D may be sufficiently large to avoid the collapse regime, that is, these disc galaxies are rotationally supported. Besides other instabilities, the ring fragmentation instability is conceptually pertinent to disc galaxies such as estimates of global star formation rates (e.g., Jog & Solomon 1984a; Kennicutt 1989; Silk 1997; Lou & Fan 1998b, 2002a, b). Having said these, we should add that the collapse regime of $D-$criterion might be relevant in early stages of galactic disc evolution. That is, slow and fast disc rotations of proto-galaxies may be responsible for eventual bifurcations into different morphologies of galactic systems. Furthermore, for central discs (with radii less than a few kiloparsecs) surrounding galactic nuclei with supermassive black holes (e.g., Lynden-Bell 1969) or for protostellar discs around central collapsed cores (e.g., Shu 1977), this D parameter may be sufficiently small to induce disc collapses. Therefore, both the collapse and ring-fragmentation regimes can be of considerable interest in various astrophysical applications.

Recently, we investigated coplanar perturbation configurations in a composite system of two gravitationally coupled stellar and gaseous SIDs, both taken to be razor thin (Lou & Shen 2003). In the ideal two-fluid formalism, we derived analytical solutions for stationary axisymmetric and logarithmic spiral configurations in a composite SID system in terms of a dimensionless rotation parameter D_s for the stellar SID*. By the analogy of a single SID (Shu et al. 2000), these stationary configurations should form marginal stability curves (see Fig. 2 and more extensive results of Lou & Shen 2003) that separate the stable region from unstable ' regions for axisymmetric disturbances. Compared with the single SID case, the stable range of D_s is reduced owing to the presence of an additional gaseous SID and can vary for different parameters.

In order to make convincing arguments for the above analogy and our physical interpretations, we perform here a time-dependent local stability analysis using the WKBJ approximation for the composite SID system (Lou & Shen 2003). In a proper parameter regime, we demonstrate the correctness of interpretations by Shu et al. (2000), and in general, we clearly show the validity of using D_s parameter to demark stable and unstable regimes. To place our analysis in relevant contexts, we also discuss how the two effective Q parameters of Elmegreen (1995) and of Jog (1996) are related to our D_s parameter when other pertinent parameters are specified.

In Section 2, we derive the time-dependent WKBJ dispersion relation for the composite SID system and define several useful dimensionless parameters. In Section 3, we present the criterion for the SID system being stable against all axisymmetric disturbances. Finally, our results are discussed and summarized in Section 4.

2 TWO-FLUID COMPOSITE SID SYSTEM

Here, we go through the basic fluid equations and derive the local dispersion relation in the WKBJ approximation (Lin & Shu 1964). For physical variables, we use either superscript or subscript s for the stellar SID and g for the gaseous SID. As both SIDs are assumed to have flat rotation curves with different speeds V_s and V_g, we express the mean angular rotation speeds of the two SIDs in the forms of

$$\Omega_s = V_s/r = a_s D_s/r, \qquad \Omega_g = V_g/r = a_g D_g/r \ , \qquad (1)$$

where a_s and a_g are the velocity dispersion of the stellar SID and the isothermal sound speed in the gaseous SID, respectively. Dimensionless rotation parameters D_s and D_g are defined as the ratios of stellar SID rotation speed to velocity dispersion and gaseous SID rotation speed to sound

* Subscript $_s$ indicates the relevance to the stellar SID.

speed, respectively. The two epicyclic frequencies are

$$\begin{aligned}\kappa_s &\equiv \{(2\Omega_s/r)[d(r^2\Omega_s)/dr]\}^{1/2} = \sqrt{2}\Omega_s \ ,\\ \kappa_g &\equiv \{(2\Omega_g/r)[d(r^2\Omega_g)/dr]\}^{1/2} = \sqrt{2}\Omega_g \ .\end{aligned} \tag{2}$$

Note that κ_s and κ_g are different in general.

By the polytropic assumption for the surface mass density and the two-dimensional pressure (e.g., Binney & Tremaine 1987), the fluid equations for the stellar disc, in the cylindrical coordinates (r, φ, z) at the $z = 0$ plane, are

$$\frac{\partial \Sigma^s}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma^s u^s) + \frac{1}{r^2}\frac{\partial}{\partial \varphi}(\Sigma^s j^s) = 0 \ , \tag{3}$$

$$\frac{\partial u^s}{\partial t} + u^s\frac{\partial u^s}{\partial r} + \frac{j^s}{r^2}\frac{\partial u^s}{\partial \varphi} - \frac{j^{s2}}{r^3} = -\frac{1}{\Sigma^s}\frac{\partial}{\partial r}(a_s^2\Sigma^s) - \frac{\partial \phi}{\partial r} \ , \tag{4}$$

$$\frac{\partial j^s}{\partial t} + u^s\frac{\partial j^s}{\partial r} + \frac{j^s}{r^2}\frac{\partial j^s}{\partial \varphi} = -\frac{1}{\Sigma^s}\frac{\partial}{\partial \varphi}(a_s^2\Sigma^s) - \frac{\partial \phi}{\partial \varphi} \ , \tag{5}$$

where Σ^s is the stellar surface mass density, u^s is the radial component of the bulk fluid velocity, j^s is the $z-$component of the specific angular momentum, a_s is the stellar velocity dispersion (or an effective "isothermal sound speed"), $a_s^2\Sigma^s$ stands for an effective two-dimensional pressure in the polytropic approximation, and ϕ is the total gravitational potential perturbation. For the corresponding fluid equations in the gaseous disc, we simply replace the relevant superscript or subscript s with g. The two sets of fluid equations are coupled by the total gravitational potential ϕ through the Poisson integral

$$\phi(r, \varphi, t) = \oint d\psi \int_0^\infty \frac{-G\Sigma(r', \psi, t)r'dr'}{[r'^2 + r^2 - 2rr'\cos(\psi - \varphi)]^{1/2}} \ , \tag{6}$$

where $\Sigma = \Sigma^s + \Sigma^g$ is the total surface mass density. Here the stellar and gaseous SIDs interact mainly through the mutual gravity on large scales (Jog & Solomon 1984a,b; Bertin & Romeo 1988; Romeo 1992; Elmegreen 1995; Jog 1996; Lou & Fan 1998b, 2000a,b; Lou & Shen 2003).

Using the above equations for the steady background rotational equilibrium (indicated by an explicit subscript 0) with $u_0^s = u_0^g = 0$, $\Omega_s = j_0^s/r^2$, and $\Omega_g = j_0^g/r^2$, we obtain

$$\begin{aligned}\Sigma_0^s &= a_s^2(1 + D_s^2)/[2\pi Gr(1 + \delta)] \ ,\\ \Sigma_0^g &= a_g^2(1 + D_g^2)\delta/[2\pi Gr(1 + \delta)] \ ,\end{aligned} \tag{7}$$

and an important relation†

$$a_s^2(D_s^2 + 1) = a_g^2(D_g^2 + 1) \ , \tag{8}$$

where $\delta \equiv \Sigma_0^g/\Sigma_0^s$ is the SID surface mass density ratio, $\beta \equiv a_s^2/a_g^2$ stands for the square of the ratio of the stellar velocity dispersion to the gas sound speed. In our analysis, we specify values δ and β to characterize different composite SID systems. For late-type spiral galaxies, gas materials are less than the stellar mass with $\delta < 1$. For young proto disc galaxies, gas materials may exceed stellar mass in general with $\delta > 1$. Thus, in our analysis and computations, both cases of $\delta < 1$ and $\delta \geq 1$ are considered. As the stellar velocity dispersion is usually larger than the gas sound speed ($a_s^2 > a_g^2$), we then have $\beta > 1$. When $\beta = 1$, we have $D_s = D_g$ by condition (8), which means the two SIDs may be treated as a single SID (Lou & Shen 2003).

For small perturbations denoted by subscripts 1, we obtain the following linearized equations

$$\frac{\partial \Sigma_1^s}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma_0^s u_1^s) + \Omega_s\frac{\partial \Sigma_1^s}{\partial \varphi} + \frac{\Sigma_0^s}{r^2}\frac{\partial j_1^s}{\partial \varphi} = 0 \ , \tag{9}$$

$$\frac{\partial u_1^s}{\partial t} + \Omega_s\frac{\partial u_1^s}{\partial \varphi} - 2\Omega_s\frac{j_1^s}{r} = -\frac{\partial}{\partial r}\left(a_s^2\frac{\Sigma_1^s}{\Sigma_0^s} + \phi_1\right) \ , \tag{10}$$

$$\frac{\partial j_1^s}{\partial t} + r\frac{\kappa_s^2}{2\Omega_s}u_1^s + \Omega_s\frac{\partial j_1^s}{\partial \varphi} = -\frac{\partial}{\partial \varphi}\left(a_s^2\frac{\Sigma_1^s}{\Sigma_0^s} + \phi_1\right) \tag{11}$$

for the stellar SID, and the corresponding equations for the gaseous SID with subscript or superscript s replaced by g in equations (9)−(11). The total gravitational potential perturbation is

$$\phi_1(r, \varphi, t) = \oint d\psi \int_0^\infty \frac{-G(\Sigma_1^s + \Sigma_1^g)r'dr'}{[r'^2 + r^2 - 2rr'\cos(\psi - \varphi)]^{1/2}} \ . \tag{12}$$

In the usual WKBJ approximation, we write the coplanar axisymmetric perturbations with the periodic dependence of $\exp(ikr + i\omega t)$ where k is the radial wavenumber and ω is the angular frequency. With this dependence in the perturbation equations, we obtain the local WKBJ dispersion relation for the composite SID system in the form of

$$\begin{aligned}&(\omega^2 - \kappa_s^2 - k^2a_s^2 + 2\pi G|k|\Sigma_0^s)\\ &\quad\times(\omega^2 - \kappa_g^2 - k^2a_g^2 + 2\pi G|k|\Sigma_0^g)\\ &\quad-(2\pi G|k|\Sigma_0^s)(2\pi G|k|\Sigma_0^g) = 0 \ .\end{aligned} \tag{13}$$

Jog & Solomon (1984a) have previously derived the similar dispersion relation with $V_s = V_g$ and thus $\kappa_s = \kappa_g$. The stability analyses of Elmegreen (1995) and Jog (1996) start with the same dispersion relation of Jog & Solomon (1984a). The WKBJ approach of Jog & Solomon (1984a) is generally applicable to any locally prescribed properties of a two-fluid disc system. In earlier theoretical studies including that of Jog & Solomon (1984a), it is usually taken that the rotation speeds of the two fluid discs are equal. Our model of two coupled SIDs here is self consistent globally with a central singularity and with different SID rotation speeds in general. When one applies the local WKBJ analysis to our composite SID system, the procedure is identical to that of Jog & Solomon (1984a) but with different local SID speeds V_s and V_g and hence different epicyclic frequencies κ_s and κ_g. Therefore, our dispersion (13) appears strikingly similar to that of Jog & Solomon (1984a) yet with different κ_s and κ_g. In short, if one prescribes two different disc speeds and thus two different epicyclic frequencies in the analysis of Jog

† This implies a close relation among the two SID rotation speeds, the stellar velocity dispersion and the gas sound speed. In this aspect, it is different from the usual local prescription of a composite disc system (e.g., Jog & Solomon 1984a, b; Bertin & Romeo 1988; Elmegreen 1995; Jog 1996; Lou & Fan 1998b). Once the ratio of a_s^2 to a_g^2 is known, the rotation speeds of the composite SID system can be determined through that of the stellar disc, i.e. D_s^2, as the two rotation speeds are coupled dynamically. In the limiting regime of $a \ll V$ and $\beta \gg 1$, V_s and V_g are comparable with V_g slightly larger by an amount of $\sim (a_s^2 - a_g^2)/(2V_s)$. In this sense, the assumption $\kappa_s \approx \kappa_g$ is reasonable.

& Solomon (1984a), the resulting dispersion relation should be our equation (13).

In our analysis, the rotation speeds of the stellar and the gaseous SIDs differ in general. In particular, since a_s^2 is usually higher than a_g^2, it follows from equations (1) and (8) that $\Omega_s < \Omega_g$. That is, the stellar disc rotates somewhat slower than the gaseous disc does. Such phenomena have been observed in the solar neighborhood as well as in external galaxies (K. C. Freeman, 2003, private communications). This is related to the so-called *asymmetric drift* phenomena (e.g., Mihalas & Binney 1981) and can be understood in terms of the Jeans equations (Jeans 1919) originally derived by Maxwell from the collisionless Boltzmann equation. In essence, random stellar velocity dispersions produce a pressure-like effect such that mean circular motions become slower (Binney & Tremaine 1987). In our treatment, this effect of stellar velocity dispersions is modelled in the polytropic approximation (see eqns (2) and (8)). Biermann & Davis (1960) discussed the possibility that the mean rotation speeds of the stars and gas in the Galaxy may be different and reached the opposite conclusion that a gas disc should rotate slower than a stellar disc does. However, their analysis based on a virial theorem that excludes the effect of stellar velocity dispersions. In other words, the stress tensor due to stellar velocity dispersions in the Jeans equation has been ignored in their considerations.

3 AXISYMMETRIC STABILITY ANALYSIS

We study the axisymmetric stability problem based on the WKBJ dispersion relation. Similar to previous analysis (Toomre 1964; Elmegreen 1995; Jog 1996; Lou & Fan 1998b), we also derive an effective Q parameter. This $Q_{\rm eff}$ is expected to depend on D_s^2 once δ and β are known.

For the convenience of analysis, we define notations

$$\begin{aligned} H_1 &\equiv \kappa_s^2 + k^2 a_s^2 - 2\pi G|k|\Sigma_0^s \ , \\ H_2 &\equiv \kappa_g^2 + k^2 a_g^2 - 2\pi G|k|\Sigma_0^g \ , \\ G_1 &\equiv 2\pi G|k|\Sigma_0^s \ , \\ G_2 &\equiv 2\pi G|k|\Sigma_0^g \ , \end{aligned} \tag{14}$$

to express dispersion relation (13) as an explicit quadratic equation in terms of ω^2, namely

$$\omega^4 - (H_1+H_2)\omega^2 + (H_1H_2 - G_1G_2) = 0 \ . \tag{15}$$

The two roots[‡] ω_+^2 and ω_-^2 of equation (15) are

$$\begin{aligned} \omega_\pm^2(k) = \frac{1}{2}\{&(H_1+H_2) \\ &\pm [(H_1+H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2}\} \ . \end{aligned} \tag{16}$$

The ω_+^2 root is always positive, which can be readily proven.[§]

[‡] The determinant $\Delta \equiv (H_1 - H_2)^2 + 4G_1G_2 \geq 0$.

[§] This conclusion holds true if $H_1 + H_2 \geq 0$. If $H_1 + H_2 < 0$, then $H_1H_2 - G_1G_2 < 0$ and the conclusion still holds.

We naturally focus on the ω_-^2 root, namely

$$\begin{aligned} \omega_-^2(k) = \frac{1}{2}\{&(H_1+H_2) \\ &- [(H_1+H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2}\} \ , \end{aligned} \tag{17}$$

to search for stable conditions that make the minimum of $\omega_-^2(k) \geq 0$ for all k.

By expanding the $\omega_-^2(k)$ root in terms of definition (14), we obtain an expression involving parameters κ_s^2, κ_g^2, a_s^2, a_g^2, Σ_0^s, Σ_0^g, and k. For a composite SID system with power-law surface mass densities, flat rotation curves, and the equilibrium properties of κ_s, κ_g, Σ_0^s and Σ_0^g, equation (17) can also be expressed in terms of four parameters a_s^2, D_s^2, δ and β, namely

$$\omega_-^2(K) = \frac{a_s^2}{2r^2}[A_2K^2 + A_1K + A_0 - \wp^{1/2}] \ , \tag{18}$$

where $K \equiv |k|r$ is the dimensionless local radial wavenumber and the relevant coefficients are defined by

$$\begin{aligned} A_2 &\equiv 1 + 1/\beta \ , \\ A_1 &\equiv -(1 + D_s^2) \ , \\ A_0 &\equiv 4D_s^2 + 2(1 - 1/\beta) \ , \end{aligned} \tag{19}$$

$$\wp \equiv B_4K^4 + B_3K^3 + B_2K^2 + B_1K + B_0 \ , \tag{20}$$

with

$$\begin{aligned} B_4 &\equiv (1 - 1/\beta)^2 \ , \\ B_3 &\equiv 2(1 + D_s^2)(1 - 1/\beta)(\delta - 1)/(1 + \delta) \ , \\ B_2 &\equiv (D_s^2 - 1 + 2/\beta)(D_s^2 + 3 - 2/\beta) \ , \\ B_1 &\equiv 4(1 + D_s^2)(1 - 1/\beta)(1 - \delta)/(1 + \delta) \ , \\ B_0 &\equiv 4(1 - 1/\beta)^2 \ . \end{aligned} \tag{21}$$

As a_s^2 has been taken out as a common factor of equation (18), the most relevant part is simply

$$A_2K^2 + A_1K + A_0 - \wp^{1/2} \tag{22}$$

that involves the stellar rotation parameter D_s^2 and the two ratios δ and β. We need to determine the value of $K_{\rm min}$ at which $\omega_-^2(K)$ takes the minimum value, in terms of D_s^2. We then substitute this $K_{\rm min}$ into expression (22) and derive the condition for D_s^2 that makes this minimal $\omega_{\rm min}^2 \geq 0$.

3.1 The D_s-criterion in the WKBJ regime

To demonstrate the stability properties unambiguously, and to confirm our previous interpretations for the marginal stability curves in a composite SID system (Lou & Shen 2003), we first present some numerical results that contain the same physics and are simple enough to be understood.

According to equation (18), ω_-^2 is a function of rotation parameter D_s^2 and wavenumber K. We then use K as the horizontal axis and D_s^2 as the vertical axis to plot contours of ω_-^2 numerically. A specific example of $\delta = 0.2$ and $\beta = 10$ is shown in Fig. 1. The shaded regions are where ω_-^2 takes on negative values and the blank region is where ω_-^2 takes on positive values. In this manner, we clearly obtain marginal stability curves (solid lines) along which $\omega_-^2 = 0$.

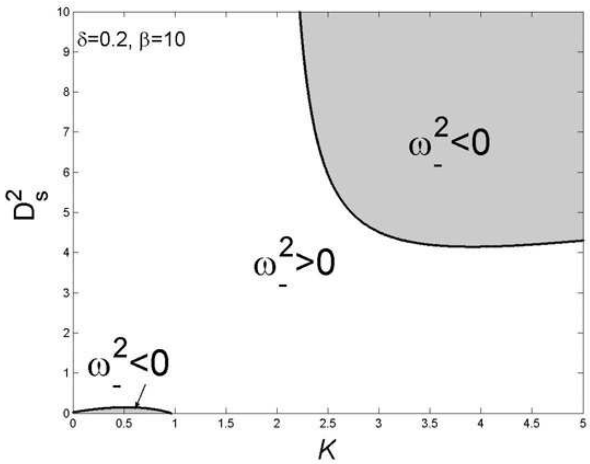

Figure 1. Contours of ω_-^2 as a function of K and D_s^2 with $\delta = 0.2$ and $\beta = 10$. The $\omega_-^2 > 0$ region is blank and the $\omega_-^2 < 0$ regions are shaded.

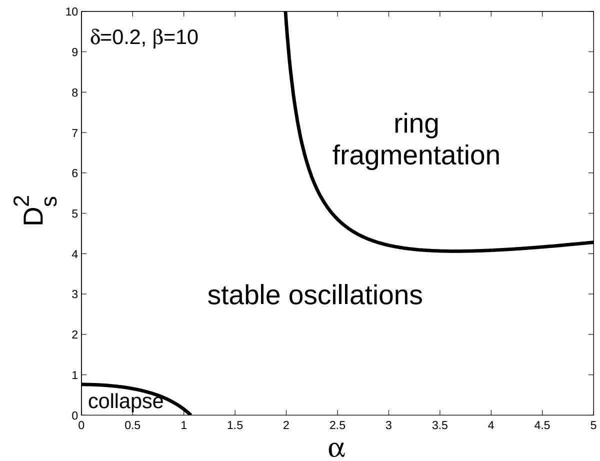

Figure 2. The marginal stability curves of D_s^2 versus the dimensionless effective radial wavenumber α for $m = 0$, $\delta = 0.2$, and $\beta = 10$. While δ remains small, a larger β lowers the ring fragmentation boundary. It is then easier for the SID system to become unstable in the form of ring fragmentation but with a larger α.

Physically, the shaded regions of negative ω_-^2 values are unstable, while the blank region of positive ω_-^2 values is stable. From such contour plots of ω_-^2, one can readily determine the specific range of stellar rotation parameter D_s^2 such that the composite SID system is stable against axisymmetric disturbances. We therefore confirm our previous interpretations for the marginal stability curves for a composite SID system (Lou & Shen 2003). For a comparison, Fig. 2 shows the corresponding marginal stability curves with $\delta = 0.2$ and $\beta = 10$ that we derived previously (Lou & Shen 2003) as the stationary axisymmetric perturbation configuration where α is a dimensionless effective radial wavenumber (see also Shu et al. 2000). The apparent difference in Figs. 1 and 2, mainly in the collapse regions, arises because the results of Fig. 2 are exact perturbation solutions without using the WKBJ approximation.

Now in the WKBJ approximation, the stable range of D_s^2 can be read off from Fig. 1. For the case of $\delta = 0.2$ and $\beta = 10$, the composite SID system is stable from $D_s^2 = 0.1193$ at $K = 0.5121$ to $D_s^2 = 4.1500$ at $K = 3.9237$. In comparison with the single SID case (see Fig. 2 of Shu et al. 2000), this stable range of D_s^2 diminishes. With other sets of δ and β parameters, we can derive similar diagrams of Fig. 2 as have been done in our recent work, for examples, Figs. $7-10$ of Lou & Shen (2003). In the WKBJ approximation, we show variations with δ and β in Figs. 3 and 4. When δ is larger, the system tends more likely to be unstable for small β. In reference to Fig. 10 of Lou & Shen (2003) with $\delta = 1$ and $\beta = 30$, we plot contours of ω_-^2 in Fig. 5 in the WKBJ approximation. In this case, the stable range of D_s^2 disappears, that is, the composite SID system cannot be stable against axisymmetric disturbances. We note again that the collapse region with small wavenumber deviates from the exact result of Fig. 10 of Lou & Shen (2003), due to the obvious limitation of the WKBJ approximation. For the unstable region of large wavenumber, which we labelled as ring fragmentation previously (Lou & Shen 2003), Fig. 5 here and Fig. 10 of Lou & Shen (2003) show good correspondence as expected. For the purpose of comparison, Table 1 lists several sets of δ and β and the corresponding stable range of D_s^2, selected from our stationary perturbation configurations studied earlier (see Figs. 5-10 of Lou & Shen 2003), using both the WKBJ approximation here and the marginal stability curves of Lou & Shen (2003).

It is not surprising that there exist one lower limit and one upper limit to bracket the stable range of D_s^2. We now provide interpretations for the single SID case. In the usual WKBJ analyses, the surface density Σ is prescribed and hence independent of the epicyclic frequency κ. In our global SID formalism, the surface mass density and the epicyclic frequency are coupled through the rotation parameter D. Both Σ and κ decrease with decreasing D but with different rates. Specifically, the surface mass density Σ is proportional to $1 + D^2$ (see eq. (7)) while κ is proportional to D (see eq. (2)). For either large or small enough D, the ratio of κ to Σ may become less than the critical value (sound speed is constant by the isothermal assumption), that is, the Q parameter becomes less than unity for instabilities.

We stress that ring fragmentation instabilities occurring at large wavenumbers are familiar in the usual local WKBJ analysis (Safronov 1960; Toomre 1964). While collapse instabilities at small wavenumbers are known recently for the SID system (Shu et al. 2000; Lou 2002; Lou & Shen 2003). The physical nature of the new "collapse" regime is fundamentally due to the Jeans instability when the radial perturbation scale becomes sufficiently large.¶ By the conservation

¶ One might wonder how a large-scale disc Jeans instability can be revealed in a local WKBJ analysis although somewhat crudely. Perhaps, the best analogy is the WKBJ dispersion relation for spiral density waves in a thin rotating disc (Lin & Shu 1966, 1968). This WKBJ dispersion relation is quadratic in the radial wavenumber $|k|$ and can be solved for a larger $|k|$ and a smaller

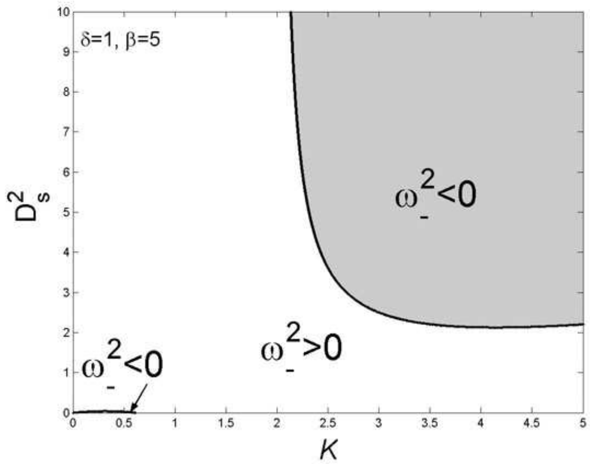

Figure 3. A contour plot of ω_-^2 as a function of K and D_s^2 with $\delta = 1$ and $\beta = 5$. The $\omega_-^2 > 0$ region is blank and the $\omega_-^2 < 0$ region is shaded.

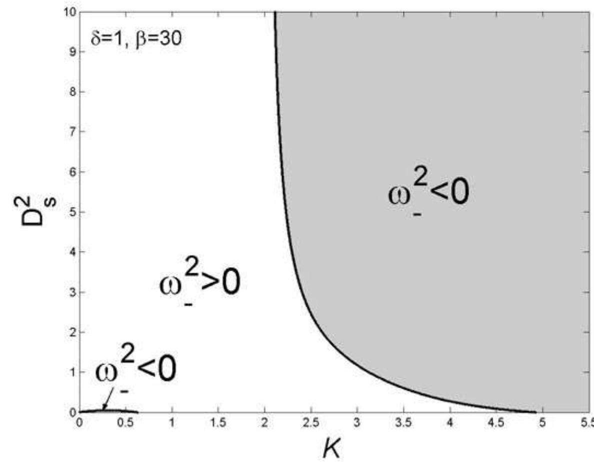

Figure 5. A contour plot of ω_-^2 as a function of K and D_s^2 with $\delta = 1$ and $\beta = 30$. The $\omega_-^2 > 0$ region is blank and the $\omega_-^2 < 0$ region is shaded.

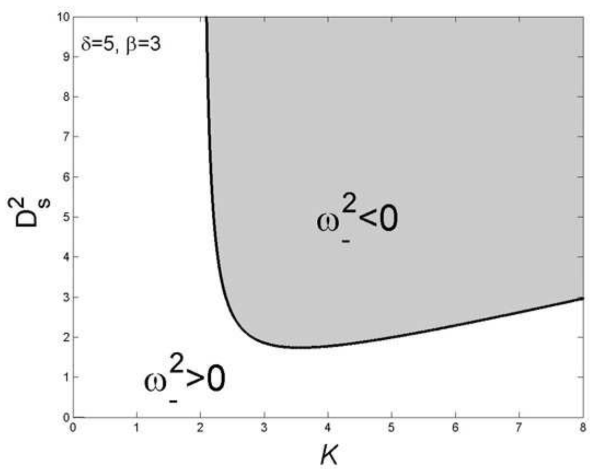

Figure 4. A contour plot of ω_-^2 as a function of K and D_s^2 with $\delta = 5$ and $\beta = 3$. The $\omega_-^2 > 0$ region is blank and the $\omega_-^2 < 0$ region is shaded.

of angular momentum, SID rotation plays a stabilizing role to modify the onset of Jeans collapse (e.g., Chandrasekhar 1961). Hence, there appears a critical D_s^2 below which exists the collapse regime. Of course, too large a D_s^2 causes the other type of ring fragmentation instabilities to set in. For most disc galaxies currently observed, the rotation parameter D_s^2 is typically large enough to avoid such collapses.

Elsewhere, the collapse regime studied here may have relevant astrophysical applications, e.g., in circumnuclear discs around nuclei of galaxies and protostellar discs around protostellar cores etc., where D_s^2 may be small enough (either a high velocity dispersion a_s or a low rotation speed V_s) to initiate collapses during certain phases of disc system evolution. In fact, for a low rotation speed V_s, we note from the perspective of evolution, that those proto disc galaxies of low rotation speeds in earlier epochs, will Jeans collapse and lead to other morphologies, while those proto disc galaxies of sufficiently fast rotation speeds in earlier epochs will gradually evolve into disc galaxies we observe today. Meanwhile, ring fragmentation instabilities are thought to be relevant to global star formation in a disc galaxy – another important aspect of galactic evolution.

The addition of a gaseous SID to a stellar SID will decrease the overall stability against axisymmetric disturbances at any wavelengths in general, that is, the stable range of D_s^2 shrinks. However, for the two types of instabilities, namely, the collapse instability at small wavenumbers and the ring-fragmentation instability at relatively large wavenumbers, the gravitational coupling of the two SIDs play different roles. For instance, a composite SID system tends more likely to fall into ring fragmentations but less likely to collapse. For either larger δ or larger β, this tendency becomes more apparent.

This trend can be understood physically. For the ring-fragmentation regime, suppose the stellar SID by itself alone is initially stable and a gaseous SID component is added to the stellar SID system in a dynamically consistent manner. The gaseous SID may be either stable or unstable by itself alone. By itself, the gaseous SID tends to be more unstable if the sound speed a_g becomes smaller or Σ_0^g becomes larger according to the definition of the Q parameter (valid for the ring fragmentation instability), which means larger β or larger δ, respectively. Thus, with either β or δ or both being larger, a composite SID system tends to be more vulnerable

$|k|$ corresponding to short- and long-wave branches respectively (e.g., Binney & Tremaine 1987). The former is naturally consistent with the WKBJ approximation, while the latter, inconsistent with the WKBJ approximation, turns out to be a necessary key ingredient in the swing process (Goldreich & Lynden-Bell 1965; Toomre 1981; Fan & Lou 1997) as revealed by analytical and numerical analyses. In the end, one needs to perform global analyses to justify the WKBJ hint for instabilities of relatively large scales.

δ	β	lower limit of D_s^2	higher limit of D_s^2
-	1	0.1716(0.9261)	5.8284(5.6259)
0.2	1.5	0.1350(0.8406)	5.3974(5.2117)
	5	0.1207(0.7705)	4.4782(4.3558)
	10	0.1193(0.7602)	4.1500(4.0611)
	30	0.1185(0.7539)	3.8062(3.7670)
1	1.5	0.0596(0.6556)	4.5945(4.4352)
	5	0.0411(0.4478)	2.1275(2.0853)
	10	0.0401(0.4221)	1.0947(1.0834)
	30	-(-)	-(-)
5	3	0.0039(0.1418)	1.7397(1.6718)
10	5	0.0011(-)	0.5696(0.5317)

Table 1. The overall stable ranges of D_s^2 against axisymmetric disturbances at any wavelengths for different sets of δ and β. The values in parentheses are derived by Lou & Shen (2003) without the WKBJ approximation and are preferred to describe the collapse region. The overall stable range of D_s^2 appears to decrease when either δ or β increases. The stable range of D_s^2 may disappear for a case like $\delta = 1$ and $\beta = 30$. When $\beta = 1$, the problem is independent of δ, that is, δ can be arbitrary (Lou & Shen 2003).

to ring fragmentations. The interpretation for the collapse regime is associated with the dynamical coupling of surface mass density Σ and epicyclic frequency κ that leads to the following conclusion, namely, a lower velocity dispersion of the gas component seems to prevent collapse. This should be understood from condition (8): for larger β, the rotation speed V_g will exceeds V_s by a larger margin and helps to prevent an overall SID collapse.

As already noted, the WKBJ approximation becomes worse in the quantitative sense when dealing with small wavenumbers where the collapse regime exists. But the stability properties of a SID can be qualitatively understood through the WKBJ analysis. The exact perturbation analysis of our recent work (Lou & Shen 2003) has shown that the collapse regime diminishes with increasing values of δ and β. Therefore, once the interpretations are justified physically, the exact procedure of Lou & Shen (2003) should be adopted to identify the relevant stable range of D_s^2 of a composite SID. As much as Shu et al. (2000) have done for a single SID analytically, Lou & Shen (2003) did the same for a composite SID system with exact solutions to the Poisson integral. The stationary axisymmetric perturbation configurations were derived to determine the marginal stability curves. These marginal stability curves have been proven to have the same physical interpretation as the WKBJ marginal curves obtained here.

3.2 Effective Q parameters

It is natural to extend the concept of the Q parameter for a single disc to an effective Q parameter for a composite disc system for the purpose of understanding axisymmetric stability properties. In the following, we discuss effective Q parameters for a composite SID system in reference to the works of Elmegreen (1995) and Jog (1996). There are several points worth noting in our formulation. First, κ_s and κ_g are related to each other but are allowed to be different in general. Secondly, the background surface mass densities are related to SID rotation speeds through the polytropic approximation and the steady background rotational equilibrium. For the convenience of notations, the effective Q parameters introduced by Elmegreen (1995) and by Jog (1996) are denoted by Q_E and Q_J, respectively.

3.2.1 The Q_E parameter of Elmegreen

Starting from equation (18) or (22), we define an effective Q_E parameter in a composite SID system following the procedure of Elmegreen (1995) but with two different epicyclic frequencies κ_s and κ_g. The minimum of ω_-^2 is given by

$$\begin{aligned}\omega_{-min}^2 &= \frac{a_s^2}{2r^2}[A_2K_{min}^2 + A_1K_{min} + A_0 - \wp^{1/2}]\\ &= \frac{a_s^2}{2r^2}(\wp^{1/2} - A_2K_{min}^2 - A_1K_{min})(Q_E^2 - 1)\ ,\end{aligned} \tag{23}$$

where

$$Q_E^2 \equiv \frac{A_0}{\wp^{1/2} - A_2K_{min}^2 - A_1K_{min}} \tag{24}$$

and $\wp$ takes on the value at $K = K_{min}$. As K_{min} depends on D_s^2, δ and β, so does Q_E^2. For specified parameters δ and β of a composite SID system, the parameter Q_E^2, giving the criterion of axisymmetric stability (i.e. stable when $Q_E^2 > 1$), corresponds to the rotation parameter D_s^2 of the stellar disc. One purpose of the present analysis is to establish the correspondence between the axisymmetric stability condition and the marginal stability curves of axisymmetric stationary perturbation configuration derived recently by Lou & Shen (2003) for a composite SID system. The pertinent results for a single SID can be found in Shu et al. (2000).

Our task now is to find the solution K_{min}. As $\kappa_s \neq \kappa_g$ in general, our computation turns out to be somewhat more complicated or involved than that of Elmegreen (1995) who took $\kappa_s = \kappa_g$. It becomes more difficult to solve this algebraic problem of order higher than the third analytically. However, if one multiplies both sides of equation (17) by the positive root ω_+^2, a simple expression appears

$$\mathcal{W} \equiv \omega_+^2\omega_-^2 = H_1H_2 - G_1G_2\ . \tag{25}$$

Instead of minimizing ω_-^2, we solve for K_c in terms of D_s^2 at which $\mathcal{W} \equiv \omega_+^2\omega_-^2$ attains the minimum value. This K_c must satisfy the following equation

$$\frac{d\mathcal{W}}{dK} = \frac{d(H_1H_2 - G_1G_2)}{dK} = 0\ . \tag{26}$$

By substituting H_1, H_2, G_1 and G_2 into equation (26) and regarding δ and β as specified parameters, equation (26) can be reduced to a cubic equation of K involving D_s^2, namely,

$$K^3 + aK^2 + bK + c = 0\ , \tag{27}$$

where

$$a = -3(D_s^2 + 1)(1 + \beta\delta)/[4(1 + \delta)]\ ,$$

$$b = D_s^2(\beta+1)+\beta-1\ ,$$

$$c = -(D_s^2+1)[\beta D_s^2(1+\delta)+\beta-1]/[2(1+\delta)]\ .$$

In most cases ‖, equation (27) has only one real solution in the form of

$$K_c = (x-q/2)^{1/3}+(-x-q/2)^{1/3}-a/3\ , \qquad (28)$$

where $x=(q^2/4+p^3/27)^{1/2}$, $p=b-a^2/3$ and $q=2a^3/27-ab/3+c$. For a single real root, it is required that $q^2/4+p^3/27>0$. Through numerical computations, we find that this condition is met for most pairs of δ and β.

We then regard this K_c as an estimator⋆⋆ for K_{min} to determine Q_E^2. As already known in our analysis (Lou & Shen 2003), when the ratio β is equal to 1, the properties of a composite SID system have something in common with those of a single SID. Moreover, the stationary axisymmetric configuration becomes the same as a single SID. We expect that the present problem should reduce to a single SID case when $\beta=1$. We consider below the special case of $\beta=1$.

With $\beta=1$ in equation (27), K_c can be determined in terms of D_s^2 without involving δ at all. We then use K_c as K_{min} in the definition of Q_E^2, namely, condition (24). One can readily verify that $Q_E^2>1$ for $3-2\sqrt{2}<D_s^2<3+2\sqrt{2}$; $Q_E^2<1$ for $D_s^2>3+2\sqrt{2}$ or $D_s^2<3-2\sqrt{2}$; and $Q_E^2=1$ for $D_s^2=3\pm2\sqrt{2}$. The range of D_s^2 that makes $Q_E^2>1$ is $0.1716<D_s^2<5.8284$, which is exactly the same as the Q estimator for the single SID of Shu et al. (2000). Based on estimates of Q parameter alone, Shu et al. (2000) made physical interpretations for their axisymmetric stationary perturbation solution curves for a single SID. According to their interpretation, the maximum $D^2=0.9320$ of the collapse branch and the minimum $D^2=5.410$ of the ring-fragmentation branch (see Fig. 2 of Shu et al. 2000) encompass the required range of D^2 for stability against axisymmetric disturbances. As the formulation of Shu et al. (2000) is exact in contrast to the WKBJ approximation, the stable range of D^2 slightly deviates from that determined by the effective Q_E parameter. And for the upper bound of D^2, with a larger wavenumber (or a smaller wavelength), the two approaches lead to consistent results as expected.

In our recent analysis (Lou & Shen 2003), we obtained curves for stationary axisymmetric perturbations given various combinations of δ and β for a composite SID system. These curves differ from those for a single SID and the stable range of D_s^2 varies for different combinations of δ and β. It is found that this stable range of D_s^2 may be significantly reduced for certain values of δ and β, as shown in Fig. 7 − 9 of Lou & Shen (2003) and in the analysis of Section 3.1. Moreover, when δ and β are sufficiently large, the stable range of D_s^2 may no longer exist, as shown in Fig. 10 of Lou & Shen (2003) and Fig. 5 here. For these cases, a composite SID system becomes vulnerable to axisymmetric instabilities.

‖ When $\kappa_s=\kappa_g$, Bertin & Romeo (1988) give the proof that as long as $1/\beta>0.0294$ and $\delta>0.172$, there exists only one minimum of ω_-^2. When there are more than one local minima for ω_-^2, one should choose K_c for the smallest minimum of ω_-^2.

⋆⋆ While K_c is obtained as the value at which ω_-^2 multiplying ω_+^2 reaches the minimum. It is valid to use K_c to determine whether $Q_E^2>1$ or $Q_E^2<1$ by realizing that ω_+^2 remains positive and the critical condition $Q_E^2=1$ is equivalent to $\omega_+^2\omega_-^2=0$.

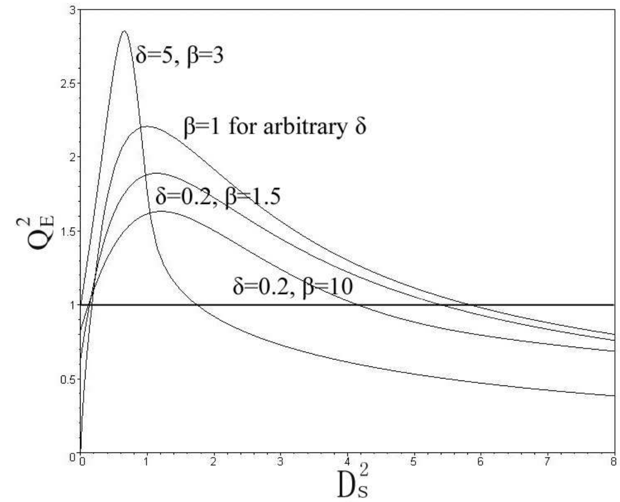

Figure 6. Curves of Q_E^2 versus D_s^2 which intersect the horizontal line $Q_E^2=1$. For each curve, the two points of intersection at $Q_E^2=1$ give the stable range of D_s^2 for a composite SID against axisymmetric disturbances. The $\beta=1$ case gives the same result of a single SID.

For $\delta=0.2$ and $\beta=10$, as shown in Fig. 6 of Lou & Shen (2003) and also Fig. 2 of this paper, we search for the approximate stable range of D_s^2 with the WKBJ dispersion relation (18) for a composite SID system. By using the Q_E^2 defined by equation (24), one can directly plot the curve of Q_E^2 versus D_s^2 as shown in Fig. 6, from which one can identify the approximate stable range of D_s^2 from ~ 0.1193 to ~ 4.1500 where $Q_E^2=1$. Several other curves with different δ and β are also shown in Fig. 6.

One thing that should be emphasized is that the Q_E^2 parameter as defined by equation (24) must be positive, otherwise it may occasionally lead to incorrect conclusions regarding the sign of ω_-^2 as defined by equation (23). More detailed investigations indicate that it is only applicable to cases where a composite SID system can be stabilized in a proper range of D_s^2. For special cases that cannot be stabilized at all, the Q_E^2 thus defined may not be relevant.

3.2.2 The Q_J parameter of Jog

The Q_E parameter can be derived by solving cubic equation (27) for K. It may happen that equation (27) gives three real solutions. In order to obtain an analytical form of K_{min}, one must identify the absolute minimum of ω_-^2. This procedure can be cumbersome. Alternatively, Jog (1996) used a seminumerical way to define another effective Q parameter, referred to as Q_J here, that is different from Q_E defined by equation (24).

According to solution (17), ω_-^2 will be positive or negative if $H_1H_2-G_1G_2>0$ or <0. The critical condition of neutral stability is thus $H_1H_2-G_1G_2=0$, which can be

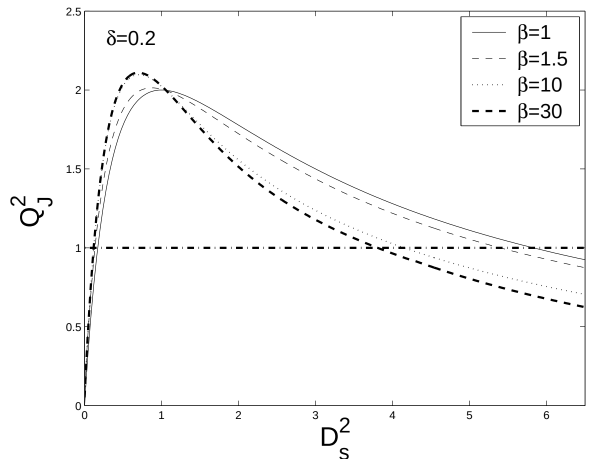

Figure 7. Curves of Q_J^2 as function of D_s^2 with $\delta = 0.2$ but for different β values. The stable ranges of D_s^2 are determined by the two points of intersection with the horizontal line $Q_J^2 = 1$. Stability corresponds to $Q_J^2 > 1$ when D_s^2 falls between the two intersection points. The two intersection points move to the left as β increases.

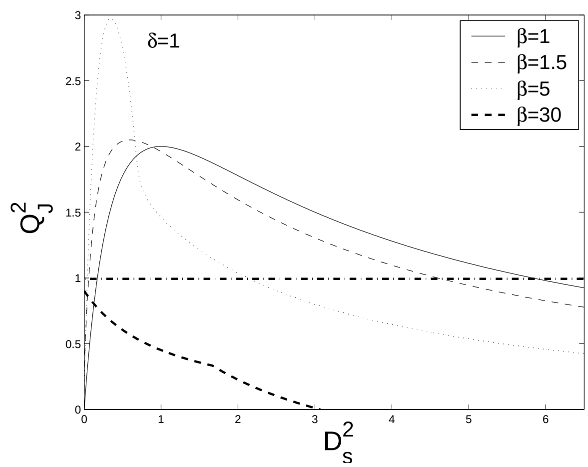

Figure 8. Curves of Q_J^2 as function of D_s^2 with $\delta = 0.2$ but for different β values. These curves intersect with the horizontal line $Q_J^2 = 1$. For a larger δ than that of Fig. 7, the two points of intersection move much faster as β increases. And the stable range of D_s^2 no longer exists when β is sufficiently large.

cast into the form of

$$\frac{2\pi Gk\Sigma_0^s}{\kappa_s^2 + k^2 a_s^2} + \frac{2\pi Gk\Sigma_0^g}{\kappa_g^2 + k^2 a_g^2} = 1 \ .$$

One can now define a function $\mathcal{F}$ such that

$$\begin{aligned} \mathcal{F} \equiv & \frac{2\pi Gk\Sigma_0^s}{\kappa_s^2 + k^2 a_s^2} + \frac{2\pi Gk\Sigma_0^g}{\kappa_g^2 + k^2 a_g^2} \\ = & \frac{K(D_s^2+1)/(1+\delta)}{2D_s^2 + K^2} + \frac{K\beta(D_s^2+1)\delta/(1+\delta)}{2[\beta(D_s^2+1)-1] + K^2} \ , \end{aligned} \tag{29}$$

where expressions (2), (7), (8) and $K \equiv |k|r$ are used. We then search for K_{min} at which a composite SID system becomes hardest to be stabilized, that is, when ω_-^2 reaches the minimum. The effective Q parameter, Q_J, can thus be defined as

$$\begin{aligned} \frac{2}{1+Q_J^2} \equiv & \frac{K_{min}(D_s^2+1)/(1+\delta)}{2D_s^2 + K_{min}^2} \\ & + \frac{K_{min}\beta(D_s^2+1)\delta/(1+\delta)}{2[\beta(D_s^2+1)-1] + K_{min}^2} \end{aligned} \tag{30}$$

(see eqns. (5) and (6) of Jog 1996). It follows that $Q_J^2 < 1$ or > 1 correspond to instability or stability, respectively. Therefore, for a given set of δ, β and D_s^2, one can numerically determine the value of K_{min}. By inserting this K_{min} into equation (30), one obtains the value of Q_J^2 for any given set of $\{\delta, \beta, D_s^2\}$. It is then possible to explore relevant parameter regimes of interest. Jog (1996) introduced three parameters, namely, the two Toomre Q parameters for each disc and the surface mass density ratio of the two discs, to determine Q_J^2. Equivalently, we use three dimensionless parameters δ, β and D_s^2 instead. Figs. 7 and 8 show variations of Q_J^2 with D_s^2. Similar to Q_E^2, the value of $Q_J^2 > 1$ when D_s^2 falls in the same ranges. When δ is fixed and β increases, the left and right bounds of the stable D_s^2 range move towards left together, while the width of the stable range appears to decrease. We have revealed the same trend of variations in our recent work (Lou & Shen 2003). It is possible for the left bound to disappear when β is sufficiently large, and it is easier to achieve this when δ becomes larger. For a sufficiently large δ, the increase of β may completely suppress the stable range of D_s^2 as shown in Fig. 5 for $\delta = 1$ and $\beta = 30$.

In comparison to Q_E, the Q_J definition (30) is valid for all parameter regimes and avoids improper situations for unusual δ and β values. However, to find the value of Q_J^2 one must perform numerical exploration for each given D_s^2, while for Q_E^2, the procedure is analytical as long as there exists a stable range of D_s^2. Regardless, the critical values of D_s^2 found by Q_E^2 and Q_J^2 are equivalent.

3.3 Composite partial SID system

In disc galaxies, there are overwhelming evidence for the existence of massive dark matter halos as inferred from more or less flat rotation curves. To mimic the gravitational effect of a dark matter halo, we include gravity terms $\partial\Phi/\partial r$ and $\partial\Phi/\partial\varphi$ in the radial and azimuthal momentum equations (4) and (5), respectively, where Φ is an axisymmetric gravitational potential due to the dark matter halo. It is convenient to introduce a dimensionless parameter $F \equiv \phi/(\phi + \Phi)$ for the fraction of the disc potential relative to the total potential (e.g., Syer & Tremaine 1996; Shu et al. 2000; Lou 2002). The background rotational equilibrium is thus modified by this additional Φ accordingly. As before, we write $\Omega_s = a_s D_s/r$, $\Omega_g = a_g D_g/r$, and $\kappa_s = \sqrt{2}\Omega_s$, $\kappa_g = \sqrt{2}\Omega_g$. The equilibrium surface mass densities now become

$$\Sigma_0^s = \frac{a_s^2(1+D_s^2)F}{2\pi Gr(1+\delta)}, \qquad \Sigma_0^g = \frac{a_g^2(1+D_g^2)F\delta}{2\pi Gr(1+\delta)} \ ,$$

where $0 \le F \le 1$. In our perturbation analysis, the dynamic response of this axisymmetric massive dark matter halo to coplanar perturbations in a composite SID system is

ignored†† (Syer & Tremaine 1996; Shu et al. 2000; Lou 2002). We thus derive similar perturbation equations as those of a full SID case ($F = 1$) and consequently the similar dispersion relation as equation (13) yet with modified equilibrium properties. Now following the same procedure of WKBJ perturbation analysis described in Section 3.1, we can plot contours of ω_-^2 as a function of rotation parameter D_s^2 and wavenumber K for a composite partial SID system. The case of $F = 1$ corresponds to a composite full SID system that has been studied in Section 3. When F becomes less than 1 (i.e., a composite partial SID system), which means the fraction of the dark matter halo increases, the stable range of D_s^2 becomes enlarged, as shown in Figs. 9 and 10 for $\delta = 0.2$ and $\beta = 10$. From these contour plots, it is clear that the introduction of a dark matter halo tends to stabilize a composite partial SID system. For late-type disc galaxies, one may take $F = 0.1$ or smaller. Such composite partial SID systems are stable against axisymmetric disturbances in a wide range of D_s^2. Moreover, those composite full SID systems that are unstable may be stabilized by the presence of a dark matter halo, as shown by the example of Fig. 11 for $\delta = 1$, $\beta = 30$ and $F = 0.5$ (this case is unstable for a composite full SID system).

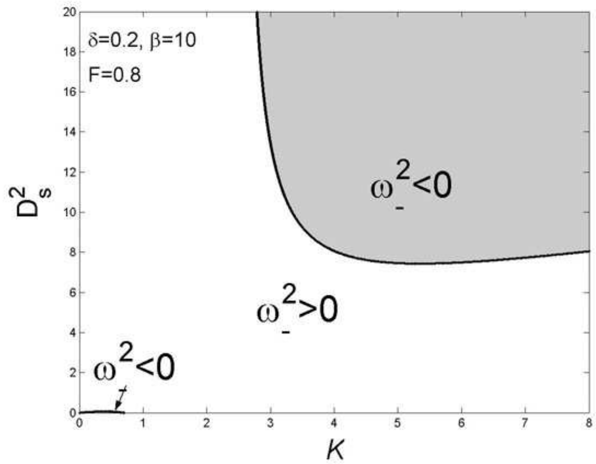

Figure 9. Contours of ω_-^2 as function of stellar rotation parameter D_s^2 and wavenumber K for a composite partial SID system with $\delta = 0.2$, $\beta = 10$ and $F = 0.8$.

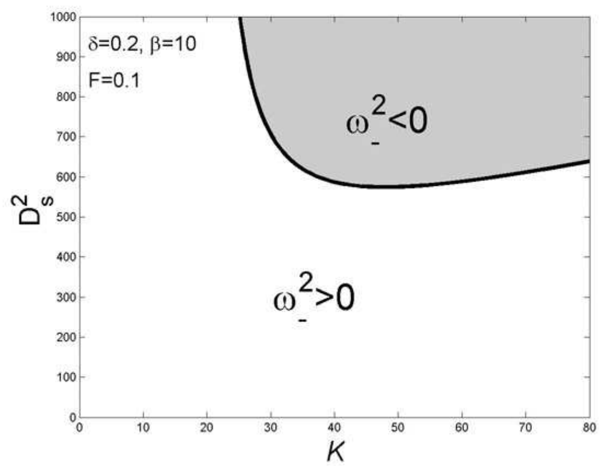

Figure 10. Contours of ω_-^2 as function of stellar rotation parameter D_s^2 and wavenumber K for a composite partial SID system with $\delta = 0.2$, $\beta = 10$ and $F = 0.1$.

4 DISCUSSIONS AND SUMMARY

Our model of a composite (full or partial) SID system is an idealization of any actual disc systems, such as disc galaxies, circumnuclear discs surrounding nuclei of galaxies, or protostellar discs. One key feature of the SID model is the flat rotation curve. One major deficiency in galactic application is the central singularity that sometimes even gives rise to conflicting theoretical interpretations (see Shu et al. 2000). The isothermality is yet another simplifying approximation. In galactic applications, the common wisdom is to introduce a bulge around the center to avoid the central singularity. By a local WKBJ stability analysis of axisymmetric perturbations, this model can be utilized as usual with qualifications. We now discuss the WKBJ results in the context of our own Galaxy, the Milky Way, for which the necessary observational data are available, such as the gas fraction, the stellar and the gaseous velocity dispersions and the epicyclic frequency, and so forth.

There are several inadequacies of our model. First, the isothermal assumption implying constant a_s and a_g throughout the disc system is a gross simplification; the velocity dispersion a_s in the Milky Way decreases with increasing radius (e.g., Lewis & Freeman 1989). Secondly, by the polytropic approximation, the surface mass densities are characterized by power-law $\propto r^{-1}$ profiles, which appear not to be the case of the Milky Way (e.g., Caldwell & Ostriker 1981). Nevertheless, to estimate local stability properties of a disc portion, we may take $a_s = 50$ km s^{-1}, $V_s = 220$ km s^{-1}, $\delta = 0.1$ and flat rotation speed $V_s = 220$ km s^{-1} around 4 kpc from the center of our Galaxy. For a massive dark matter halo such that $F = 0.1$, the relevant parameters to determine the

†† Strictly speaking, gravitational perturbation coupling between a thin, less massive, rotating disc and a more massive dark matter halo together with proper matching and boundary conditions should lead to an increasing number of modes that may or may not be stable. We simplify by noting numerical simulations have indicated that a typical galactic dark matter halo has a relatively high velocity dispersion of the order of several hundred kilometers per second. The rationale of the simplification is that perturbation coupling between those in the SID system and those in the dark matter halo becomes weak for a high velocity dispersion in the dark matter halo. For example, for a non-rotating spherical dark matter halo, effective "acoustic" and "gravity" modes (i.e., $p-$modes and $g-$modes in the parlance of stellar oscillations; Lou 1995) can exist within; these $p-$ and $g-$modes are expected to be Landau damped in a collisionless system. However, the stability of "interface modes" between a thin rotating disc and a dark matter halo remains to be investigated.

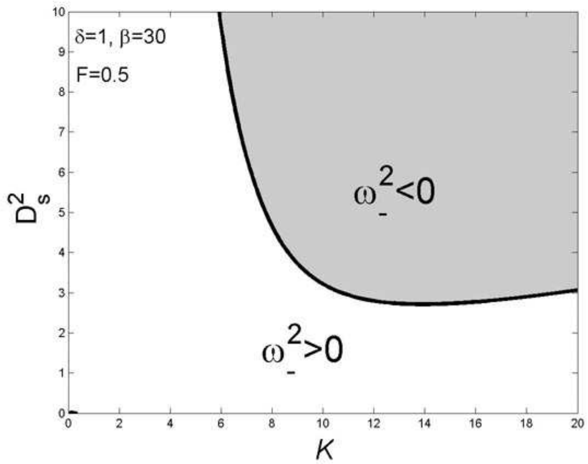

Figure 11. Contours of ω_-^2 as function of stellar rotation parameter D_s^2 and wavenumber K for a composite partial SID system with $\delta = 1$, $\beta = 30$ and $F = 0.5$.

effective Q parameter for a composite partial SID system are $\delta = 0.1$, $\beta = 50$, $D_s^2 = 20$ and $F = 0.1$. Following the definition of Jog (1996), the value of $Q_J \gg 1$. While for full SIDs with $F = 1$, we get $Q_J = 0.17$ which means local instability. Obviously, the dark matter halo plays the key role in stabilizing the system.

The composite partial SID model with power-law $\propto r^{-1}$ surface mass densities may not describe the Milky Way well. But for those disc galaxies with flat rotation curves and approximate power-law $\propto r^{-1}$ surface mass densities, our model should offer a sensible local criterion for axisymmetric instability. For example, given radial variations of velocity dispersions and gas mass fraction, we may deduce locally unstable zones.

For circumnuclear discs around nuclei of galaxies, the stellar velocity dispersion a_s can be as high as several hundred kilometers per second along the line of sight (e.g., Whitmore et al. 1985; McElroy 1994); this may then lead to small D_s^2. Within such a parameter regime, the collapse instability may set in during a certain phase of system evolution. For instance, in the case of Fig. 2 with $\delta = 0.2$ and $\beta = 10$, we choose $V_s = 200$ km s^{-1} and $a_s = 300$ km s^{-1} in a circumnuclear disc region of a disc galaxy. The resulting $D_s^2 = 0.44$ would give rise to collapse and the composite SID system might eventually evolve into a bulge of high velocity dispersion.

Another possible collapse situation of interest corresponds to small D_s^2 for sufficiently low disc rotation speed V. Potential applications include but not limited to protostellar discs in the context of star formation. We here briefly discuss several qualitative aspects. For a protostellar disc system, the two disc components here may be identified with the relatively "hot" gas disc and the relatively "cool" dust disc. Interactions of radiation fields from the central protostar with cool dusts in the disc lead to infrared emissions. In our idealized treatment, the gas and dust discs are treated as two gravitationally coupled thin SIDs. This simple model treatment may need to be complemented by other processes of coupling such as collisions between gas and dust particles. When such a composite disc system rotates at a low speed (e.g., Shu et al. 1987), the rotation parameter D could be low enough to initiate large-scale collapse. The subsequent dynamical processes will drive the composite SID system to more violent star formation activities.

So far we have investigated the axisymmetric stability problem for a composite system of gravitationally coupled stellar and gaseous SIDs using the WKBJ approximation and we now summarize the results of our analysis.

First, because the single SID case studied by Shu et al. (2000) may be regarded as the special case of a composite SID system, the results of our WKBJ analysis clearly support the physical interpretations of Shu et al. (2000) for the marginal stability curves of axisymmetric perturbations. The "ring fragmentation regime" is related to the familiar Q parameter (Safronov 1960; Toomre 1964). The presence of the Jeans "collapse regime" can be traced to the SID model where self-gravity, effective pressure, and SID rotation compete with each other.

Secondly, the recent study of Lou & Shen (2003) generalizes that of Shu et al. (2000) and describes exact perturbation solutions in a composite SID system. We obtained marginal stability curves for axisymmetric perturbations that are qualitatively similar to those of Shu et al. (2000) and that depend on additional dimensionless parameters. It is natural to extend the interpretations of Shu et al. (2000). Through the WKBJ analysis here, we now firmly establish the presence of two regimes of "ring fragmentation" and "collapse" in a composite SID system. It is fairly straightforward to apply our exact $D-$criterion for a composite SID system, not only for disc galaxies, but also for other disc systems including circumnuclear discs and protostellar discs etc.

Thirdly, in the WKBJ approximation, it is also possible to relate our $D-$criterion to the Q_E criterion similar to that of Elmegreen (1995). Because κ_s and κ_g are different in general, our WKBJ treatment is not the same as that of Elmegreen (1995).

Fourthly, also in the WKBJ approximation, we relate our $D-$criterion to the Q_J criterion as defined by Jog (1996) but with $\kappa_s \neq \kappa_g$ in general. We find that the Q_J criterion is fairly robust in the WKBJ regime.

Finally, we further consider the axisymmetric stability of a composite partial SID system to include the gravitational effect from an axisymmetric dark matter halo. The stabilizing effect of the dark matter halo is apparent.

ACKNOWLEDGMENTS

We thank the referee C. J. Jog for comments and suggestions. This research has been supported in part by the ASCI Center for Astrophysical Thermonuclear Flashes at the University of Chicago under Department of Energy contract

B341495, by the Special Funds for Major State Basic Science Research Projects of China, by the Tsinghua Center for Astrophysics, by the Collaborative Research Fund from the NSF of China (NSFC) for Young Outstanding Overseas Chinese Scholars (NSFC 10028306) at the National Astronomical Observatory, Chinese Academy of Sciences, and by the Yangtze Endowment from the Ministry of Education through the Tsinghua University. Affiliated institutions of Y.Q.L. share the contribution.

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